

Non- D_1 dense Banach subalgebras of $c_0(\mathbb{N})$

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Abstract

I construct dense Banach subalgebras A of the null sequence algebra $c_0(\mathbb{N})$ which are spectral invariant, but do not satisfy the D_1 -condition $\|ab\|_A \leq K(\|a\|_\infty\|b\|_A + \|a\|_A\|b\|_\infty)$, for all $a, b \in A$. The sequences in A vanish in a skewed manner with respect to an unbounded function $\sigma: \mathbb{N} \rightarrow [1, \infty)$.

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1 Introduction

The theory of differential geometry on a C^* -algebra (or noncommutative space) [Co, 1994] requires the use of “differentiable structures” for these noncommutative spaces. Such differentiable structures have usually been provided by a dense subalgebra of smooth functions A for which the K -theory $K_*(A)$ is the same as the K -theory of the C^* -algebra $K_*(B)$. We say that A is *spectral invariant* in B if the quasi-invertible elements of A are precisely those elements of A which are quasi-invertible in B . Recall that for $a, b \in A$, $a \circ b = a + b - ab$, and b is a quasi-inverse for a if and only if $a \circ b = b \circ a = 0$ [Pa, 1994], Definition 2.1.1. The spectral invariance of a Fréchet algebra A in B implies closure under holomorphic functional

calculus, which in turn implies an isomorphism of K -theory $K_*(A) \cong K_*(B)$ [Co, 94], Chapter 3, Appendix C.

Let $c_0(\mathbb{N})$ be the C^* -algebra of complex-valued vanishing sequences, or null sequences, on the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$, with pointwise multiplication and involution. For $n \in \mathbb{N}$, let $e_n: \mathbb{N} \rightarrow \{0, 1\}$ be the unit step function at n , where $e_n(k) = 1$ if $k = n$ and $e_n(k) = 0$ for $k \in \mathbb{N} \setminus \{n\}$. Let $c_f(\mathbb{N})$ denote the linear span of $\{e_n\}_{n=0}^\infty$, which is a dense ideal in $c_0(\mathbb{N})$. The following result shows it is relatively easy for a dense subalgebra of $c_0(\mathbb{N})$ to be spectral invariant.

Theorem 1.1. *Let A be a dense Banach subalgebra of the null sequence algebra $B = c_0(\mathbb{N})$. Assume that the subalgebra of finite support functions $\mathcal{A} = c_f(\mathbb{N})$ is dense in A . Then A is spectral invariant in B .*

Proof: Let $a \in A \setminus qinv(A)$. Note that $A(1 - a)$ is a proper ideal in A , and moreover $a + A(1 - a) \cap qinv(A) = \emptyset$. Since A is a Banach algebra, $qinv(A)$ is open. Hence $A(1 - a)$ cannot be dense in A , and $\mathcal{A} \not\subseteq A(1 - a)$. Since $A(1 - a)$ is a linear space, there is some $n_0 \in \mathbb{N}$ for which $e_{n_0} \notin A(1 - a)$. Since $A(1 - a)$ is an \mathcal{A} -module, every element must vanish at n_0 , and $a(n_0) = 1$. This implies that a is not quasi-invertible in B . \square

A dense Banach subalgebra A of a C^* -algebra B is a D_1 -subalgebra if for

some constant $K_A > 0$ we have

$$\|ab\|_A \leq K_A \left\{ \|a\|_A \|b\|_B + \|a\|_B \|b\|_A \right\}, \quad (1)$$

for all $a, b \in A$ [Ki, 1994][Bh, 2016]. D_1 implies spectral invariance [Ki, 1994], Theorem 5 and Lemma 4, which raises the question: is every dense subalgebra of the null sequence algebra, which satisfies the hypotheses of Theorem 1.1, also D_1 ? The purpose of this paper is to provide a counterexample. To this end, we think of $c_0(\mathbb{N})$ as a direct sum

$$\bigoplus_{n=0}^{\infty} \mathbb{C}^2, \quad (2)$$

where the isomorphism is given by

$$f \in c_0(\mathbb{N}) \longleftrightarrow \left\{ (f(0), f(1)), \dots (f(2n), f(2n+1)), \dots \right\}, \quad (3)$$

and \mathbb{C}^2 denotes the two-dimensional commutative C^* -algebra, with coordinate-wise multiplication and involution. If A is a dense subalgebra, D_1 is satisfied along each \mathbb{C}^2 -summand, because it is finite dimensional. In §2, we construct submultiplicative norms $\|\cdot\|_n$ on the n th copy of \mathbb{C}^2 , which make the D_1 -constants K_n become unbounded as n increases. In §3, we define the Banach algebra norm $\|\cdot\|_A$ as the sup of these \mathbb{C}^2 -norms to construct the counterexample.

2 Some norms on \mathbb{C}^2

Let $r \in \mathbb{R}$ and let σ be a constant in $[1, \infty)$. Let $\|\vec{v}\|_{\max} = \max\{|x|, |y|\}$, $\vec{v} = (x, y) \in \mathbb{C}^2$, denote the C^* -norm on \mathbb{C}^2 . Define a seminorm on \mathbb{C}^2 by

$$\|\vec{v}\|_{r,\sigma} = \|T_{r,\sigma}\vec{v}\|_{\max} = \max\{|x + ry|, \sigma|y - rx|\}, \quad (4)$$

where $T_{r,\sigma}$ is the 2×2 -matrix

$$\begin{pmatrix} 1 & r \\ -\sigma r & \sigma \end{pmatrix}. \quad (5)$$

Note that $\Delta = \det(T_{r,\sigma}) = \sigma(1 + r^2) > 0$, so $T_{r,\sigma}$ is invertible, and $\|\cdot\|_{r,\sigma}$ is a norm on \mathbb{C}^2 .

We want to find the smallest constant $D > 0$ which satisfies $\|\vec{v}\|_{\max} \leq D\|\vec{v}\|_{r,\sigma}$ for all $\vec{v} \in \mathbb{C}^2$. Then D is the norm of

$$T_{r,\sigma}^{-1} = \frac{1}{\Delta} \begin{pmatrix} \sigma & -r \\ \sigma r & 1 \end{pmatrix} \quad (6)$$

as an operator on \mathbb{C}^2 ,

$$\begin{aligned} D &= \|T_{r,\sigma}^{-1}\|_{\text{op}} = \frac{1}{\sigma(1 + r^2)} \max\{\sigma + |r|, \sigma|r| + 1\} \\ &= \frac{1}{1 + r^2} \max\{1 + |r|/\sigma, |r| + 1/\sigma\} \\ &\leq \frac{1}{1 + r^2} \max\{1 + |r|, |r| + 1\} \quad \text{since } \sigma \geq 1 \\ &= \frac{1 + |r|}{1 + r^2}. \end{aligned} \quad (7)$$

As r ranges over all real numbers, the last expression in (7) is bounded by 1.21.

Next we want to find a constant $C > 0$ which satisfies $\|\vec{v}_1 * \vec{v}_2\|_{r,\sigma} \leq C\|\vec{v}_1\|_{r,\sigma}\|\vec{v}_2\|_{r,\sigma}$, for all $\vec{v}_1, \vec{v}_2 \in \mathbb{C}^2$, where $\vec{v}_1 * \vec{v}_2$ denotes pointwise multiplication. This is equivalent to finding the norm of the operator $\vec{u}_1 \otimes \vec{u}_2 \mapsto T_{r,\sigma}(T_{r,\sigma}^{-1}\vec{u}_1 * T_{r,\sigma}^{-1}\vec{u}_2)$ from $\mathbb{C}^2 \otimes \mathbb{C}^2$ to \mathbb{C}^2 . The operator is $\frac{1}{\Delta^2}$ times

$$\begin{aligned}
& \begin{pmatrix} 1 & r \\ -\sigma r & \sigma \end{pmatrix} \left(\begin{pmatrix} \sigma & -r \\ \sigma r & 1 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} * \begin{pmatrix} \sigma & -r \\ \sigma r & 1 \end{pmatrix} \begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix} \right) \\
&= \begin{pmatrix} 1 & r \\ -\sigma r & \sigma \end{pmatrix} \begin{pmatrix} (\sigma u_{11} - r u_{12})(\sigma u_{21} - r u_{22}) \\ (\sigma r u_{11} + u_{12})(\sigma r u_{21} + u_{22}) \end{pmatrix} \\
&= \begin{pmatrix} 1 & r \\ -\sigma r & \sigma \end{pmatrix} \begin{pmatrix} \sigma^2 & -r\sigma & -r\sigma & r^2 \\ \sigma^2 r^2 & \sigma r & \sigma r & 1 \end{pmatrix} \begin{pmatrix} u_{11}u_{21} \\ u_{11}u_{22} \\ u_{12}u_{21} \\ u_{12}u_{22} \end{pmatrix} \\
&= \begin{pmatrix} \sigma^2(1+r^3) & \sigma(r^2-r) & \sigma(r^2-r) & r^2+r \\ \sigma^3(r^2-r) & \sigma^2(r^2+r) & \sigma^2(r^2+r) & \sigma(1-r^3) \end{pmatrix} \begin{pmatrix} u_{11}u_{21} \\ u_{11}u_{22} \\ u_{12}u_{21} \\ u_{12}u_{22} \end{pmatrix} \quad (8)
\end{aligned}$$

So the operator norm is

$$\begin{aligned}
C &= \frac{\sigma \max \left\{ \frac{|1+r^3|}{\sigma} + \frac{2|r^2-r|}{\sigma^2} + \frac{|r^2+r|}{\sigma^3}, |r^2-r| + \frac{2|r^2+r|}{\sigma} + \frac{|1-r^3|}{\sigma^2} \right\}}{(1+r^2)^2} \\
&\leq \frac{\sigma \max \left\{ |1+r^3| + 2|r^2-r| + |r^2+r|, |r^2-r| + 2|r^2+r| + |1-r^3| \right\}}{(1+r^2)^2},
\end{aligned} \tag{9}$$

where the second step used $\sigma \leq 1$. As r ranges over all real numbers, the last expression in (9) is bounded by 2σ .

It follows that the norm $\|\cdot\|_{r,\sigma}$, when multiplied by the constant 2σ , is submultiplicative. The new constant D gets divided by 2σ , and is bounded above by $1/\sigma$.

3 The Banach algebras $A_{r,\sigma}$

In this section, we pass from the case of a single copy of \mathbb{C}^2 (§2) to an infinite direct sum of copies of \mathbb{C}^2 . Let $r \in \mathbb{R}$ and let σ be any proper function from \mathbb{N} to $[1, \infty)$. Define an infinite matrix $S_{r,\sigma}$ as the direct sum of 2×2 matrices

$$S_{r,\sigma} = \bigoplus_{n=0}^{\infty} 2\sigma(n)T_{r,\sigma(n)}, \tag{10}$$

where each $T_{r,\sigma(n)}$ is defined as in (5). Regarding $S_{r,\sigma}$ as an operator on the space of all sequences of complex numbers, let $A_{r,\sigma}$ be the subspace of $c_0(\mathbb{N})$

which is mapped into $c_0(\mathbb{N})$. By [Wi, 1984], Theorem 4.3.1, $A_{r,\sigma}$ is complete for the norm

$$\|f\|_{r,\sigma} = \|S_{r,\sigma}f\|_\infty = \sup_{n=0}^\infty \left\{ 2\sigma(n) \|(f(2n), f(2n+1))\|_{r,\sigma(n)} \right\}, \quad (11)$$

where each $\|\cdot\|_{r,\sigma(n)}$ is defined as in (4), and where by the final remarks of §2, $\|fg\|_{r,\sigma} \leq \|f\|_{r,\sigma}\|g\|_{r,\sigma}$ and $\|f\|_\infty \leq \|f\|_{r,\sigma}$, for $f, g \in A_{r,\sigma}$. Since $S_{r,\sigma}f \in c_0(\mathbb{N})$ for $f \in A_{r,\sigma}$,

$$\lim_{n \rightarrow \infty} 2\sigma(n) \|(f(2n), f(2n+1))\|_{r,\sigma(n)} = 0. \quad (12)$$

It follows that $c_f(\mathbb{N})$ is dense in $A_{r,\sigma}$, and we can apply Theorem 1.1 to see that $A_{r,\sigma}$ is spectral invariant in $c_0(\mathbb{N})$.

Theorem 3.1. *The Banach subalgebra $A_{r,\sigma}$ of $c_0(\mathbb{N})$ is not D_1 for $r \in \mathbb{R} \setminus \{0, 1\}$.*

Proof: For $n \in \mathbb{N}$, define $a_n \in A_{r,\sigma}$ by

$$a_n = \left(\vec{0}, \vec{0}, \dots, (1, r), \vec{0}, \dots \right), \quad (13)$$

where the nonzero entry is in the n th copy of \mathbb{C}^2 , and $\vec{0} = (0, 0)$ denotes the zero vector in \mathbb{C}^2 . Then

$$\begin{aligned} \|a_n\|_{r,\sigma} &= 2\sigma(n)(1 + r^2), \\ \|a_n^2\|_{r,\sigma} &= 2\sigma(n) \max(|1 + r^3|, \sigma(n)|r^2 - r|), \\ \|a_n\|_\infty &= \max(1, |r|). \end{aligned} \quad (14)$$

If $K > 0$ were a constant satisfying the D_1 -condition (1) for $A_{\sigma,r}$ in $c_0(\mathbb{N})$, then

$$\begin{aligned} K &\geq \frac{\|a_n^2\|_{r,\sigma}}{2\|a_n\|_\infty\|a_n\|_{r,\sigma}} = \frac{\max(|1+r^3|, \sigma(n)|r^2-r|)}{2(1+r^2)\max(1,|r|)} \\ &\geq \frac{\sigma(n)|r^2-r|}{2(1+r^2)\max(1,|r|)} \end{aligned} \quad (15)$$

must hold for each $n = 0, 1, 2, \dots$. No such constant K can exist if $r \neq 1$ or 0, since σ is unbounded. \square

Remark 3.2. $A_{r,\sigma}$ is a Banach \star -algebra. The norms defined on \mathbb{C}^2 (4) and the norm on $A_{r,\sigma}$ (11) are both left unchanged by the \star -operation of pointwise complex-conjugation.

Remark 3.3. The cases $r = 0$ and $r = 1$. It can be shown that $A_{r,\sigma}$ is a D_1 -subalgebra of $c_0(\mathbb{N})$ in these cases, and is a dense Banach ideal in $c_0(\mathbb{N})$ if $r = 0$, but not if $r = 1$.

4 References

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